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NO RANDOM REALS IN COUNTABLE SUPPORT ITERATIONS*

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ABSTRACT

We prove a preservation theorem for limit steps of countable support iterations of proper forcing notions whose particular cases are preservations of the following properties on limit steps: "no random reals are added", "$\mu(\text{Random}(\mathbf{V})) \neq 1$", "no dominating reals are added", "Cohen($\mathbf{V}$) is not comeager". Consequently, countable support iterations of σ-centered forcing notions do not add random reals.

Introduction

There are many results on iterated forcing which can be understood as preservation theorems. Let us mention the following two kinds of preservation theorems. Let Φ be a property of forcing notions and let $\langle P_i, \dot{Q}_i \colon i < \alpha \rangle$ be an iterated forcing system and let P_α be a limit of this system.

(A) Assume that for each $i < \alpha$, $\Vdash_{P_i}$ "$\dot{Q}_i$ has the property Φ". Has P_α the property Φ too?

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(B) Assume that for each $i < \alpha$, P_i has the property Φ. Has P_α the property Φ too?

The properties "P does not add an unbounded real" and "P does not add a convergent series which cannot be majorized by a series from the ground model" of a forcing notion P ensure that the set of old objects will remain dominant in the generic extension. Preserving the base of the ideal of null sets or preserving the base of the ideal of meager sets are similar properties of forcing notions. Let us note that for proper forcing notions all these examples of the properties satisfy the preservation theorem of type (A). This is a consequence of a general preservation theorem proved in [4] (see also [10], Theorem XVIII.3.6).

Let us recall a problem of Judah and Shelah (see [4], Problem 0.14): Assume that $\langle P_n, \dot{Q}_n \colon n \in \omega\rangle$ is an iterated forcing system with inverse limit P_ω. Does the assumption that for every $n \in \omega$, P_n does not add Cohen reals imply that P_ω does not add Cohen reals?

Problem 0.13 of [4] is concerned with the preservation (of type (A)) of the property "no random reals are added". It is known (see [3]) that a two step iteration $P * \dot{Q}$ can add a random real over $\mathbf{V}$ although P does not add random reals over $\mathbf{V}$ and $\Vdash_P$ "$\dot{Q}$ does not add random reals over $\mathbf{V}[\dot{G}]$". However in this example the forcing notion P is not proper.

We will prove a preservation theorem of the type (B) for the property "no random reals are added" assuming all forcing notions are proper. Note that for finite support iteration of c.c.c. forcing notions even much stronger versions of such result are known to be true (see e.g. [2]). We will prove our preservation theorem in a somewhat more general context so that as a consequence we get preservation of some other properties of forcing notions (not adding a dominating real, not adding a majorant series, etc.).

The preservation theorem

Let $\langle \sqsubseteq_n : n \in \omega\rangle$ be an increasing sequence of two place relations on ${}^\omega\omega$. We let $\sqsubseteq = \bigcup_n \sqsubseteq_n$. We always think of a formula $\varphi(n, x, y)$ which defines $x \sqsubseteq_n y$. This provides a definition of relations $\sqsubseteq_n$ in all models containing the parameters of φ. We need from φ to be absolute in all transitive models (in all our examples of the use of the preservation theorem φ is even arithmetical). We also require the following to be satisfied:

(i) The formula $\varphi(n, h, x)$ is a definition of a Borel set which is a relatively closed subset of rng($\sqsubseteq$) with parameters n, h, i.e. $\{x\colon h \sqsubseteq_n x\}$ is relatively closed in rng($\sqsubseteq$) (with the same Borel code in all transitive models) for all $n \in \omega$, $h \in {}^\omega\omega$.

(ii) Whenever $A \subseteq \mathrm{dom}(\sqsubseteq)$ is countable then there is $f \in \mathrm{dom}(\sqsubseteq)$ such that for every $g \in A$ and every $n \in \omega$ there is $k \geq n$ such that

$$(\forall x)\ f \sqsubseteq_k x \rightarrow g \sqsubseteq_k x.$$

(iii) The formula

$$\psi(n) = (\forall x \in {}^\omega\omega)\ f \sqsubseteq_n x \rightarrow g \sqsubseteq_n x$$

is absolute for all transitive models containing f and g. (Note that if φ is arithmetical then ψ is Π^1_1 and so absolute.)

Note that we do not require from $\sqsubseteq$ to be transitive. In the case when $\sqsubseteq$ is transitive and reflexive the condition (ii) says somewhat more than that every countable system is $\sqsubseteq$-bounded by a single element.

We say that a real x is $\sqsubseteq$-dominating over $\mathbf{V}$ if for all $y \in \mathbf{V} \cap \mathrm{dom}(\sqsubseteq)$, $y \sqsubseteq x$.

EXAMPLES: (1) For $f, g \in {}^\omega\omega$ and $n \in \omega$ set

$$f \sqsubseteq^b_n g \text{ iff } (\forall k \geq n)\ f(k) \leq g(k), \quad \text{and} \quad f \sqsubseteq^b g \text{ iff } (\forall^\infty k)\ f(k) \leq g(k).$$

Hence, f is $\sqsubseteq^b$-dominating over $\mathbf{V}$ if and only if f is dominating over $\mathbf{V}$.

(2) Let $\mathcal{K} = \{f \in {}^\omega\mathcal{Q}_+\colon \sum_{n\in\omega} f(n) \leq 1\}$, where $\mathcal{Q}_+$ is the set of positive rationals endowed with the discrete topology. For $f, g \in \mathcal{K}$ and for $n \in \omega$ set

$$f \sqsubseteq^{\mathcal{K}}_n g \text{ iff } (\forall k \geq n)\ f(k) \leq g(k), \quad \text{and} \quad f \sqsubseteq^{\mathcal{K}} g \text{ iff } (\forall^\infty k)\ f(k) \leq g(k).$$

Hence f is a $\sqsubseteq^{\mathcal{K}}$-dominating real over $\mathbf{V}$ if and only if f majorizes all series from $\mathbf{V}$.

(3) Let $\mathcal{S} = \{f \in {}^\omega([{}^{<\omega}2]^{<\omega})\colon (\forall n \in \omega)\ f(n) \subseteq {}^n2\ \&\ \sum_{n\in\omega} |f(n)|2^{-n} \leq 1\}$ be ordered by $f \leq^* g$ iff $(\forall^\infty n)\ f(n) \subseteq g(n)$.

Every function $f \in \mathcal{S}$ represents a G_δ set $A_f = \{x \in {}^\omega2\colon (\exists^\infty k)\ x{\restriction}k \in f(k)\}$ of Lebesgue measure zero and every measure zero set is a subset of some set A_f. A variant of the Stern–Raisonnier result [8] says that every subfamily of $\mathcal{S}$ of cardinality less than add($\mathcal{N}$) (additivity of Lebesgue measure) is bounded.

The set $\mathcal{S}$ is a closed subset of the product ${}^{\omega}([{}^{<\omega}2]^{<\omega})$, where the countable set $[{}^{<\omega}2]^{<\omega}$ is endowed with the discrete topology. For $f \in \mathcal{S}$, $x \in {}^{\omega}2$ and $n \in \omega$ we define

$$f \sqsubseteq_n^r x \text{ iff } (\forall k \geq n)\ x{\restriction}k \notin f(k), \quad \text{and} \quad f \sqsubseteq^r x \text{ iff } x \notin A_f.$$

Hence a real x is $\sqsubseteq^r$-dominating over $\mathbf{V}$ if and only if x is a random over $\mathbf{V}$.

(4) Let $\Omega = {}^{<\omega}2$ and let $\mathcal{C} = \{f \in {}^{\Omega}\Omega\colon (\forall s \in \Omega)\ s \subseteq f(s)\}$. For $f, g \in \mathcal{C}$ and $n \in \omega$ let

$$f \sqsubseteq_n^{\mathcal{C}} g \text{ iff } \bigcup_{|s|\geq n} [g(s)] \subseteq \bigcup_{|s|\geq n} [f(s)], \quad \text{and} \quad f \sqsubseteq^{\mathcal{C}} g \text{ iff } (\exists n)\ f \sqsubseteq_n^{\mathcal{C}} g.$$

Clearly,

$$\{x \in {}^{\Omega}\Omega\colon f \sqsubseteq_n^{\mathcal{C}} x\} = \bigcap_{|s|\geq n} \bigcup_{n \leq m \leq g(s)} \{x \in \mathcal{C}\colon f(g(s){\restriction}m) \subseteq g(s)\}$$

is a closed subset of ${}^{\Omega}\Omega$ which is obviously homeomorphic to the Baire space.

CLAIM: *There is a $\sqsubseteq^{\mathcal{C}}$-dominating real over $\mathbf{V}$ if and only if the set Cohen($\mathbf{V}$) of Cohen reals over $\mathbf{V}$ is comeager.*

Proof: If g is $\sqsubseteq^{\mathcal{C}}$-dominating over $\mathbf{V}$ then the set $\bigcap_{n\in\omega} \bigcup_{|s|\geq n}[g(s)]$ is a comeager set of Cohen reals.

Conversely, let us assume that there is comeager set of Cohen reals. By an application of Kuratowski–Ulam theorem we can find a Cohen real over $\mathbf{V}$ such that Cohen($\mathbf{V}[c]$) is still comeager (see [7], Proposition 1.2). We will find a $\sqsubseteq^{\mathcal{C}}$-dominating real over $\mathbf{V}$.

For $s \in {}^{<\omega}2$ let $c_s \in {}^{\omega}2$ be defined as follows: $c_s(n) = s(n)$, for $n \in \mathrm{dom}(s)$, and $c_s(n) = c(n)$, for $n \geq |s|$. Let $f \in \mathcal{C} \cap \mathbf{V}$ be arbitrary. Since c_s, for $s \in {}^{<\omega}2$, are Cohen reals over $\mathbf{V}$, we can define a function $h_f\colon {}^{<\omega}2 \to \omega$, in $\mathbf{V}[c]$, so that for every $t \in {}^{<\omega}2$,

$$[c_t{\restriction}h_f(t)] \subseteq \bigcup_{|s|\geq|t|} [f(s)],$$

since the right-hand side is open dense. Since Cohen($\mathbf{V}[c]$) is comeager, there is a dominating real over $\mathbf{V}[c]$ (see [6], the proof of Theorem 1.2), and so there is a function $h\colon {}^{<\omega}2 \to \omega$ such that $(\forall s)\ |s| \leq h(s)$ and for every $f \in \mathcal{C} \cap \mathbf{V}$, $(\forall^{\infty} s)\ h_f(s) \leq h(s)$.

Define $g(s) = c_s \restriction h(s)$ for $s \in {}^{<\omega}2$. Then for every $f \in \mathcal{C} \cap \mathbf{V}$ there is $n \in \omega$ such that $h(s) \geq h_f(s)$ for all s, $|s| \geq n$, and so

$$\bigcup_{|s|\geq n} [g(s)] \subseteq \bigcup_{|s|\geq n} [c_s \restriction h_f(s)] \subseteq \bigcup_{|s|\geq n} [f(s)].$$

Therefore, g is a $\sqsubseteq^{\mathcal{C}}$-dominating real over $\mathbf{V}$. The Claim is proved. ∎

All these relations satisfy the required properties (i)–(iii). Note that all of them satisfy the following strengthening of (ii):

(ii)′ Whenever $A \subseteq \operatorname{dom}(\sqsubseteq)$ is countable then there is $f \in \operatorname{dom}(\sqsubseteq)$ such that for every $g \in A$ for all but finitely many $k \in \omega$, $(\forall x)\ f \sqsubseteq_k x \to g \sqsubseteq_k x$.

For examples (1) and (2) this is a consequence of the fact that the respective relation $\sqsubseteq$ is transitive and for every countable family there is a $\sqsubseteq$-dominating element in the range of $\sqsubseteq$. For example (3) the condition (ii)′ follows easily from the fact that every countable subfamily of S is bounded by a single element with respect to the ordering $\leq^*$. We will prove the property (ii)′ for the example (4) only.

Let $f_0, f_1, f_2, \ldots$ be a sequence of elements of $\mathcal{C}$. For arbitrary $s \in {}^{<\omega}2$, let us define $s_i \in {}^{<\omega}2$, $i \in \omega$ as follows: $s_0 = s$, $s_{i+1} = f_i(s_i)$. Let us define $f \in \mathcal{C}$ by $f(s) = s_{|s|}$. Then $f(s)$ extends all $f_i(s_i)$ for $i < |s|$ and for every n, and $i < n$,

$$\bigcup_{|s|\geq n} [f(s)] \subseteq \bigcup_{|s|\geq n} [f_i(s_i)] \subseteq \bigcup_{|t|\geq n} [f_i(t)].$$

Hence $f_i \sqsubseteq_n^{\mathcal{C}} f$ for all $n > i$ and by transitivity of the relations $\sqsubseteq_n^{\mathcal{C}}$, the property (ii)′ follows.

Note that the relations in examples (1), (2) and (4) placed into Theorem 3 will produce preservation theorems for not adding a dominating real, not adding a majorant series, and not adding a comeager set of Cohen reals, respectively. It is known (see e.g. [1]) that not adding a majorant series is equivalent to the fact that the set of random reals does not have full measure. Hence we get a preservation of the property "$\mu(\text{Random}(\mathbf{V})) \neq 1$."

The relation $\sqsubseteq^r$ (example (3)) placed into Theorem 3 will produce a preservation theorem for not adding a random real.

The following two technical lemmata will be used in the proof of Theorem 3.

LEMMA 1: *Let $P * \dot{Q}$ be an iteration of proper forcing notions. Let $p \in P$ be an (N, P)-generic condition and let $p \Vdash \dot{r} \in \dot{Q} \cap N[\dot{G}]$, where $\dot{G}$ is the canonical name for a generic filter on P. Then there is a P-name $\dot{q}$ such that $(p, \dot{q}) \leq (p, \dot{r})$ and $(p, \dot{q})$ is $(N, P * \dot{Q})$-generic.*

Proof: By hypothesis there is a maximal antichain $\{p_i : i \in I\}$ below p and P-names $\dot{r}_i \in N$ such that $p_i \Vdash \dot{r} = \dot{r}_i$. All p_i are (N, P)-generic. Now there are P-names $\dot{q}_i$ for conditions in $\dot{Q}$ such that $(p_i, \dot{q}_i) \leq (p_i, \dot{r}_i)$ and $(p_i, \dot{q}_i)$ is $(N, P * \dot{Q})$-generic (see [9], page 91). By the fullness of V^P (see [5], Lemma 18.5) there is $\dot{q} \in V^P$ such that $p_i \Vdash \dot{q} = \dot{q}_i$. Then, obviously, $(p, \dot{q}) = \bigvee\{(p_i, \dot{q}_i) : i \in I\}$ and so $(p, \dot{q})$ is $(N, P * \dot{Q})$-generic and $(p, \dot{q}) \leq (p, \dot{r})$. ∎

We will use a special case of the situation appearing in the proof of the previous lemma.

Definition: Let $p \in P$, $\dot{r} \in V^P$. We say that p has an (N, P)-evidence about the name $\dot{r}$ if there is an antichain $A \subseteq P \cap N$ such that $p \leq \bigvee A$ and for every $q \in A$ there is a P-name $\dot{r}_q \in N$ such that $q \Vdash \dot{r} = \dot{r}_q$. (Note that $p \Vdash \dot{r} \in N[\dot{G}]$.) ∎

LEMMA 2: *Let φ be a formula of the forcing language with parameters in N and let $\dot{y} \in V^P$. If p is (N, P)-generic, p has an (N, P)-evidence about $\dot{y}$ and $p \Vdash \exists x\ \varphi(x, \dot{y})$, then there is a P-name $\dot{x}$ such that $p \Vdash \varphi(\dot{x}, \dot{y})$ and p has an (N, P)-evidence about $\dot{x}$.*

Proof: Let $A \subseteq P \cap N$ be an antichain witnessing the (N, P)-evidence of p about $\dot{y}$. Hence for $r \in A$ there is a P-name $\dot{y}_r \in N$ such that $r \Vdash \dot{y} = \dot{y}_r$. Consequently, the Boolean value $a_r = ||\exists x\ \varphi(x, \dot{y}_r)||$ is in N. For $r \in A$, let $A_r \in N$ be an antichain which is maximal with the property $A_r \subseteq \{t \in P : t \leq a_r \ \&\ t \leq r\}$. $A' = \bigcup_{r \in A} A_r$ is an antichain and, by the assumption of the lemma, $p \leq \bigvee A'$. Since the antichains A_r and the values $\bigvee A_r$ are all in N and since p is (N, P)-generic, only the conditions in $A_r \cap N$, for some $r \in A$, can be compatible with p. Consequently, $p \leq \bigvee (A' \cap N)$.

Without loss of generality we can assume $A = A' \cap N$ and so $r \Vdash \exists x\ \varphi(x, \dot{y}_r)$, for $r \in A$. By the existential completeness lemma there is a P-name $\dot{x}_r \in N$ such that $r \Vdash \varphi(\dot{x}_r, \dot{y}_r)$. Define $\dot{x}$ so that $r \Vdash \dot{x} = \dot{x}_r$, for $r \in A$. Obviously, $p \Vdash \varphi(\dot{x}, \dot{y})$ and the antichain $A = A' \cap N$ witnesses the (N, P)-evidence of p about $\dot{x}$. ∎

THEOREM 3: *Let P_ω be the inverse limit of an iterated forcing system $\langle P_n, \dot{Q}_n \colon n \in \omega\rangle$. If for all $n \in \omega$, P_n is proper and it does not add a $\sqsubseteq$-dominating real then P_ω does not add a $\sqsubseteq$-dominating real.*

Proof: Let τ be a P_ω-name for a real, $p \in P_\omega$. We find $q \leq p$ and $f \in {}^\omega\omega \cap \mathbf{V}$ such that $q \Vdash_\omega f \not\sqsubseteq \tau$. Hence τ is not $\sqsubseteq$-dominating over V.

Let χ be a large enough cardinal and let $N \prec H(\chi)$ be such that $\tau, p, P_\omega \in N$. Let $f \in {}^\omega\omega \cap \mathrm{dom}(\sqsubseteq)$ be such that

$$(\forall x \in {}^\omega\omega)\,(f \sqsubseteq x \to (\forall g \in \mathrm{dom}(\sqsubseteq) \cap N)\ g \sqsubseteq x)\,.$$

By induction on $n \in \omega$ we define

(i) a P_n-name $\dot{p}_n$ for a condition in $\dot{P}_{n,\omega}$,

(ii) an (N, P_n)-generic condition $q_n \in P_n$,

such that

(iii) q_n has an (N, P_n)-evidence about $\dot{p}_n$,

(iv) $q_{n+1}{\restriction}n = q_n$, $(q_{n+1}, \dot{p}_{n+1}) \leq (q_n, \dot{p}_n)$, $(q_{n+1}, \dot{p}_{n+1}) \Vdash_\omega f \not\sqsubseteq_n \tau$.

For $n = 0$ set $\dot{p}_0 = p$, $q_0 = \emptyset$. Obviously, condition (iii) is satisfied. Let $n \in \omega$ be arbitrary and let us assume that we have constructed $\dot{p}_n$, q_n. We will find $\dot{p}_{n+1}$, q_{n+1}.

Since q_n has an (N, P_n)-evidence about $\dot{p}_n$, by Lemma 2, there is a P_n-name $\langle \dot{p}_{n,m} \colon m \in \omega\rangle$ for a decreasing sequence of conditions in $\dot{P}_{n,\omega}$ and a P_n-name τ_n for a real such that

$$q_n \Vdash_n (\forall m)(\dot{p}_{n,m+1} \leq \dot{p}_{n,m} \ \ \&\ \ \dot{p}_{n,m} \Vdash_{n,\omega} \tau{\restriction}m = \tau_n{\restriction}m)$$

and such that q_n has an (N, P_n)-evidence about $\langle \dot{p}_{n,m} \colon m \in \omega\rangle$ and τ_n.

By the hypothesis of the theorem, τ_n is not a name for a $\sqsubseteq$-dominating real, and so using Lemma 2 again, there is a P_n-name $\dot{h}$ for a real from the ground model such that $q_n \Vdash_n \dot{h} \not\sqsubseteq \tau_n$ and q_n has an (N, P_n)-evidence about $\dot{h}$. By the choice of f, using the properties (ii) and (iii) of $\sqsubseteq$, we can find a P_n-name a for an integer greater than n such that

$$(*) \qquad q_n \Vdash_n (\forall x \in {}^\omega\omega)\ f \sqsubseteq_{\dot{a}} x \to \dot{h} \sqsubseteq_{\dot{a}} x$$

and such that q_n has an (N, P_n)-evidence about $\dot{a}$. (Use an antichain A which witnesses the (N, P_n)-evidence of q_n about $\dot{h}$ such that every condition $r \in A$

decides a $g_r \in N$ for the name $\dot{h}$. By the definition of f and the property (ii) of $\sqsubseteq$, we can find a sequence $\{k_r: r \in A\}$ of integers $\geq n$ such that $(\forall x \in {}^\omega\omega)\ f \sqsubseteq_{k_r} x \rightarrow g_r \sqsubseteq_{k_r} x$. Now define $\dot{a}$ so that $r \Vdash \dot{a} = k_r$. Then the property $(*)$ holds by absoluteness of ψ.)

In particular, $q_n \Vdash_n \dot{h} \not\sqsubseteq_{\dot{a}} \tau_n$. Since $\{x: \dot{h} \not\sqsubseteq_{\dot{a}} x\}$ is a name for a relatively open set containing τ_n in V^{P_n}, using Lemma 2, there is a P_n-name $\dot{m}$ for an integer so that

$$q_n \Vdash_n [\tau_n \restriction \dot{m}] \subseteq \{x \in {}^\omega\omega: \dot{h} \not\sqsubseteq_{\dot{a}} x\}$$

and q_n has an (N, P_n)-evidence about $\dot{m}$. Then by $(*)$,

$$q_n \Vdash_n [\tau_n \restriction \dot{m}] \subseteq \{x: f \not\sqsubseteq_{\dot{a}} x\} \subseteq \{x: f \not\sqsubseteq_n x\}.$$

Now set $\dot{p}_{n+1} = \dot{p}_{n,\dot{m}} \restriction \langle n+1, \omega)$ and, using Lemma 1, choose $q_{n+1} \in P_{n+1}$ such that $q_{n+1} \restriction n = q_n$ and $q_n \Vdash_n q_{n+1}(n) \leq \dot{p}_{n,\dot{m}}(n)$. Note that q_{n+1} has an (N, P_{n+1})-evidence about $\dot{p}_{n+1}$ and $(q_{n+1}, \dot{p}_{n+1}) \leq (q_n, \dot{p}_n)$. At last

$$(q_{n+1}, \dot{p}_{n+1}) \leq (q_n, \dot{p}_{n,\dot{m}}) \Vdash_\omega \tau \in [\tau_n \restriction \dot{m}] \subseteq \{x \in {}^\omega\omega: f \not\sqsubseteq_n x\}.$$

Therefore, the conditions (iii) and (iv) are satisfied.

Now define $q = \bigcup_{n \in \omega} q_n$. Obviously, $q \Vdash_\omega f \not\sqsubseteq \tau$ and so the proof of the theorem is finished. ∎

COROLLARY 4: *Let $\langle P_n, \dot{Q}_n: n \in \omega\rangle$ be an iteration forcing system and let P_ω be the inverse limit of this system. Then*

(a) *if for all $n \in \omega$, P_n does not add random reals, then P_ω does not add random reals,*

(b) *if for all $n \in \omega$, P_n does not add dominating reals, then P_ω does not add dominating reals,*

(c) *if for all $n \in \omega$, P_n does not add a set of random reals of positive measure, then P_ω does not add a set of random reals of positive measure,*

(d) *if for all $n \in \omega$, P_n does not add a comeager set of Cohen reals, then P_ω does not add a comeager set of Cohen reals.*

THEOREM 5: *Any forcing notion obtained as a result of the countable support iteration of σ-centered forcing notions does not add random reals*

Proof: The limit case of the proof follows from Theorem 3 and the nonlimit case is a corollary of the forthcoming lemma. Let us consider the system of functions from example (3):

$$\mathcal{S} = \{f \in {}^{\omega}(\mathcal{P}({}^{<\omega}2)) \colon (\forall n \in \omega)\ f(n) \subseteq {}^{n}2\ \ \&\ \sum_{n \in \omega} \frac{|f(n)|}{2^n} \leq 1\},$$

$$f \leq^* g \quad \text{iff } (\exists m)(\forall n > m)\ f(n) \subseteq g(n).$$

If $\mathcal{F} \subseteq \mathcal{S}$, then $\mathbf{b}(\mathcal{F})$ denotes the minimal cardinality of an unbounded subset of $\mathcal{F}$ with respect to the order $\leq^*$ restricted to $\mathcal{F}$.

We say that a family $\mathcal{F} \subseteq \mathcal{S}$ is a **covering family** if the following two conditions are satisfied:

(1) $(\forall x \in {}^{\omega}2)(\exists f \in \mathcal{F})(\exists^{\infty} k)\ x{\restriction}k \in f(k)$,

(2) $\mathbf{b}(\mathcal{F}) \geq \omega_1$.

Easily it can be seen that a forcing notion P does not add random reals if and only if

$$\Vdash_P \text{"the family } \mathcal{S} \cap \mathbf{V} \text{ is covering."}$$

Hence the next preservation lemma is sufficient for the nonlimit step in the proof of the theorem.

LEMMA 6: *Let Q be a σ-centered forcing notion. Then for every covering family $\mathcal{F}$,*

$$\Vdash_Q \text{"the family } \mathcal{F} \text{ is covering."}$$

In the proof we will need this lemma.

LEMMA 7 ([11]): *Let $Q = \bigcup_n Q_n$ where each Q_n is centered. Let a be a finite set and let τ be a Q-name of a member of a, i.e. $\Vdash_Q \tau \in a$. Then for arbitrary index n there is $k \in a$ such that*

$$(\forall p \in Q_n)(\exists q \leq p)\ q \Vdash \tau = k.$$

Proof: If not, then for every $k \in a$ there is a condition $q_k \in Q_n$ such that $q_k \Vdash \tau \neq k$. Since Q_n is centered, there is a condition q stronger than all q_k and this condition forces $\tau \notin a$ which is a contradiction. ∎

Proof of Lemma 6: Let $\dot{x}$ be arbitrary Q-name of a real. For every $n \in \omega$ the set

$$T_n = \{s \in {}^{<\omega}2 \colon (\forall p \in Q_n)(\exists q \leq p)\ q \Vdash s \subseteq \dot{x}\}$$

is a subtree of the tree ${}^{<\omega}2$ and, according to Lemma 6, this is an infinite tree. Let $y_n \in {}^{\omega}2$ be an infinite branch of T_n. Since $\mathcal{F}$ is a covering family and $\mathbf{b}(\mathcal{F}) \geq \omega_1$, there is a function $f \in \mathcal{F}$ such that for all $n \in \omega$, $(\exists^\infty k \in \omega)\ y_n \restriction k \in f(k)$. We show that $\Vdash_Q (\exists^\infty k \in \omega)\ \dot{x} \restriction k \in f(k)$.

On the contrary let us assume that for some condition $p \in Q$ and for some $m \in \omega$,

$$p \Vdash (\forall k > m)\ \dot{x} \restriction k \notin f(k).$$

There is $n \in \omega$ such that $p \in Q_n$. Let $k > m$ be arbitrary such that $y_n \restriction k \in f(k)$. Now, since $y_n \restriction k \in T_n$, there is $q \leq p$ such that $q \Vdash \dot{x} \restriction k = y_n \restriction k$ and so $q \Vdash \dot{x} \restriction k \in f(k)$. This is a contradiction since $k > m$ and $q \leq p$.

The proofs of Lemma 6 and Theorem 5 are finished. ∎

Note that Lemma 6 is true also for κ-centered forcing notions for any $\kappa < \text{add}(\mathcal{N})$.

PROBLEMS: (1) *Assume $\langle P_n, \dot{Q}_n : n \in \omega \rangle$ is an iteration forcing system of random algebras with the inverse limit P_ω. Is there a random real in $\mathbf{V}^{P_\omega}$ which is a random real over all $\mathbf{V}^{P_n}$, $n \in \omega$?*

(2) *Find a two step iteration $P * \dot{Q}$ of proper forcing notions such that P does not add random reals, $\Vdash_P$ "$\dot{Q}$ does not add random reals", but $P * \dot{Q}$ adds a random real.*

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